# Solutions for Exercise Sheet 3

November 11, 2012

1. (a) $z \in \mathbb{C}$
   (b) $z \neq 0 + 2\pi k i, k \in \mathbb{Z}$
   (c) $z \neq 0 + (\frac{\pi}{2} + 2\pi k)i, k \in \mathbb{Z}$
   (d) $z \neq 2\pi k, k \in \mathbb{Z}$

2. First, notice that the function is differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. Therefore it is enough to check CR equations, and indeed:

$$\exp(x)\cos(y) = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \exp(x)\cos(y)$$

and:

$$-\exp(x)\sin(y) = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\exp(x)\sin(y)$$

3. (a) The positive numbers.
   (b) The positive numbers rotated by a constant angle.
   (c) Circles.
   (d) It is a spiral.

4. Using the definition of the sine function:

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$

The cosine function:

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}$$

And the property of the exponent:

$$\exp(z + u) = \exp(z) \times \exp(u)$$

It is a simple computation to prove the formula.

5. (a) $2\pi i$
   (b) $0$

6. (a) $|\int_{|z|=R} f(z)dz| \leq 2\pi R \cdot \sup_{z\in\mathbb{C}} |f(z)| \xrightarrow{R\to 0} 0$
   (b) It is not true. Any constant function is a counterexample.

7. Simple computation yields:

$$\int_{|z|=1} z^n dz = 0$$

   Uniform convergence allows us to change the order of summation and integration, and than all the summands are 0.

8. (a) $R = \frac{1}{3}$
   (b) $R = \frac{1}{2}$
   (c) $R = 0$
   (d) $R = 1$

9. Let $\gamma_{R,1}$ be the path that goes in straight line from $R$ to $R+i\sqrt{R}$, $\gamma_{R,2}$ be the path that goes in straight lines from $R+i\sqrt{R}$ to $R+iR$ to $-R+iR$ and than to $-R+i\sqrt{R}$ and $\gamma_{R,3}$ be the stright line between $-R+i\sqrt{R}$ to $-R$. Notice that:

$$\int_{\gamma_R} f(z)dz = \int_{\gamma_{R,1}} f(z)dz + \int_{\gamma_{R,2}} f(z)dz + \int_{\gamma_{R,3}} f(z)dz$$

   The next part is to show that each of these summands converge to 0. Indeed:

$$|\int_{\gamma_{R,1}} f(z)dz| \leq \mathit{Length} \times \mathit{Supremum} = \sqrt{R} \cdot \frac{1}{R} \xrightarrow{R\to\infty} 0$$

   The above is also true for $\gamma_{R,3}$, and for $\gamma_{R,2}$ we have:

$$|\int_{\gamma_{R,2}} f(z)dz| \leq \mathit{Length} \times \mathit{Supremum} = (4R - 2\sqrt{R}) \cdot \frac{\exp(-R)}{R} \xrightarrow{R\to\infty} 0$$